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On Faster than Nyquist Signaling: Computing the Minimum Distance

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In this paper we are concerned with the problem of computing the minimum of the L_2 norm taken over the interval $(-\delta, \delta)$, $0<\delta\leqslant 1/2$, and over all non-trivial linear combinations of the functions $\exp(2\pi in\theta)$, $n=0, 1, ...$; the coefficients in the linear combination being restricted to 0, ± 1. Denoting the minimal L_2 norm over the interval $(-\delta, \delta)$ by $I(\delta)$, it is trivial by the orthogonality of the exponentials that $I(1/2)$ is 1. The main result of the paper is to show that there is a neighborhood of half for which $I(\delta)$ is 1, despite the non-orthogonality of the exponentials in the interval $(-\delta, \delta)$. The origin of this problem, arises from certain basic problems in data communications, concerned with studying the behavior of the minimum L_2 distance between signals, when data is sent faster than the Nyquist rate over an ideal bandlimited channel. The above mentioned result shows that there is no degradation in the minimum distance for rates somewhat faster than the Nyquist rate.

1. INTRODUCTION

In this paper we are concerned with the problem of computing the minimum of the L_2 norm taken over the interval $(-\delta, \delta)$, $0<\delta\leqslant 1/2$, and over all non-trivial linear combinations of the functions $\exp(2\pi in\theta)$, $n=0, 1, ...$; the coefficients in the linear combination being restricted to 0, ± 1. The origin of this problem, which is explained more fully after we state the mathematical formulation, arises from certain basic problems in data communications, concerned with the behavior of the minimal L_2 distance between signals, when data is sent faster than the so-called Nyquist rate, over an ideal bandlimited channel.

The mathematical formulation of the problem is as follows: For $0<\delta\leqslant 1/2$, let

$$I(\delta)=\inf_{p\in E}\left(\frac{1}{2\delta}\int_{-\delta}^{\delta}|p(\theta)|^2\,d\theta\right)^{1/2},$$

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where $E = \{\sum_{k=0}^{n} \varepsilon_k e^{2\pi i k\theta} \mid n = 0, 1, ...; \varepsilon_k = 0, \pm 1, \varepsilon_0 = 1\}$. We are interested in the behavior of $I(\delta)$ for $0 < \delta \leqslant 1/2$. Note that by the orthogonality of the exponentials, $I(1/2) = 1$ and further that $I(\delta)$ tends to 0 as δ tends to 0, since for the polynomial $p(\theta) = 1 - e^{2\pi i\theta}$,

$$\lim_{\delta \to 0} \frac{1}{2\delta} \int_{-\delta}^{\delta} |p(\theta)|^2 \, d\theta = 0.$$

(More precisely $p(\theta/2\pi)$ is a trigonometric polynomial as in [7]. We simply call $p(\theta)$ a polynomial; see the notation following Theorem 3.) Since the exponentials are no longer orthogonal in $L_2(-\delta, \delta)$ for $0 < \delta < 1/2$, this raises the question [5] as to whether there is a $\delta_0 < 1/2$ such that $I(\delta) = 1$ for $\delta_0 \leqslant \delta \leqslant 1/2$. The same question can also be asked regarding $I(\delta, L)$ where the definition of $I(\delta, L)$ is exactly the same as $I(\delta)$ except that now, when defining E, the condition on ε_k is that $|\varepsilon_k| \leqslant L$ (and $\varepsilon_0 = 1$). In view of the Stone–Weierstrass theorem it is somewhat surprising that the following can be shown:

THEOREM 1. *There is a* $\delta_0(L) < 1/2$ *such that* $I(\delta, L) = 1$ *for* $\delta_0(L) \leqslant \delta \leqslant 1/2$.

Before going further, we state the origin and relevancy of the above problem to data communications. It has been known since the 1920's that Nyquist pulses

$$g(t) = \frac{\sin(\pi t/T)}{\pi t/T}$$

can be used to send data without intersymbol interference over bandlimited channels. Precisely, this means that one sends signals

$$\sum_{n=n_1}^{n_2} a_n g(t - nT)$$

if one wants to send binary data $a_n = \pm 1$ over a channel of bandwidth $1/2T$. The absence of intersymbol interference means that the peak of a pulse $g(t - nT)$ is at the zero-crossings of the other pulses $g(t - mT)$; $m \neq n$. The above facts have played a major role in the design and implementation of data transmission over the telephone network.

Now suppose we use pulses

$$g(t) = A \frac{\sin \pi t/T}{\pi t/T}$$

but send such pulses at intervals $R = 2\,\delta T$ with $0 < \delta < 1/2$ instead of $R = T$. We assume optimum processing of the received signals. We now encounter intersymbol interference and it is natural to use the minimum L_2 distance between received signals as a performance criterion. In this case it is easily seen [5] that the minimum distance can be gauged in terms of $I(\delta)$, in the case of binary data being sent. Thus $\delta = 1/2$ corresponds to the classical Nyquist rate, and the question asked earlier about whether $I(\delta) = 1$ in a neighborhood of $\delta = 1/2$ corresponds to asking whether there is a non-degradation of the minimum L_2 distance between received signals for rates of transmission somewhat faster than the Nyquist rate.

Besides the above motivation, it should be mentioned that Forney [1] and refinements by others [2, 6, 10] have shown that the bit error rate may be tightly estimated in terms of the minimum distance for high signal-to-noise ratios. It seems probable that the techniques of this paper can be used to compute the minimum distance when pulse shapes other than Nyquist pulses are used. Since the bit error rate is a basic parameter in gauging the performance of data communication channels, this would be of interest.

Returning to Theorem 1, we shall restrict ourselves to $I(\delta)$, the proofs being the same when $I(\delta, L)$ (which corresponds to multilevel signaling instead of just binary signaling) is considered. Thus we show,

THEOREM 2. *$I(\delta) = 1$ for $0.4975 \leqslant \delta \leqslant 0.5$.*

Actually a far stronger result is true (see [3]): Let $R(\theta) = \sum_{k=0}^{7} (-1)^k e^{2\pi i k\theta}$ and let $0 < v < 1/2$ be defined by $(1/2v) \int_{-v}^{v} |R(\theta)|^2 \, d\theta = 1$. Then $v = 0.401\ldots$, $I(\delta) = 1$ for $v \leqslant \delta \leqslant 1/2$, and $I(\delta) < 1$ for $\delta < v$ because $(1/2\delta) \int_{-\delta}^{\delta} |R(\theta)|^2 \, d\theta < 1$ for $\delta < v$. We prove Theorem 2 here, instead of this stronger result because Theorem 2 is essential to proving the stronger result and because the proof of the stronger result is considerably more difficult. Moreover, many of the elements in the proof of Theorem 2 are similar to those in the proof of the stronger result. Most importantly, the stronger result is a result peculiar to Nyquist pulses and the proof techniques in [3] do not seem to apply to other pulse shapes, while the techniques here seem to apply.

It is not hard to obtain upper bounds on $I(\delta)$, by numerically considering various polynomials. This was done in [5], where the problem of studying $I(\delta)$ was also proposed. Lower bounds on $I(\delta)$ are considerably harder to get and prior to the conference version of this paper [11] it was only known that $I(\delta) \neq 0$ for all $0 < \delta < 1/2$ [5].

Lastly we prove an extremal form of Theorem 2, which shows that the only reason $I(\delta) = 1$ for $\delta_0 \leqslant \delta \leqslant 1/2$ and some $\delta_0 < 1/2$ is because we have to consider the trivial polynomial $Q(\theta) = 1$:

THEOREM 3. *Given any* $1 \leqslant M < \sqrt{2}$ *there is a* $\delta_0 < 1/2$ *such that for* $\delta_0 \leqslant \delta \leqslant 1/2$ *and any polynomial* $Q(\theta) = \sum_{k=0}^{n} \varepsilon_k e^{2\pi i k\theta}$ *with* $\varepsilon_0 = 1$, $\varepsilon_k = 0, \pm 1$, *and* $Q(\theta) \neq 1$,

$$\left(\frac{1}{2\delta}\int_{-\delta}^{\delta} |Q(\theta)|^2\, d\theta\right)^{1/2} \geqslant M.$$

Our notation is standard other than noted below. A polynomial in this paper shall mean $\sum_{k=0}^{n} \varepsilon_k e^{2\pi i k\theta}$ where $\varepsilon_k = 0, \pm 1$, $\varepsilon_0 \neq 0$. Also $e^{2\pi i\theta}$ *is denoted* $e(\theta)$ and for a function g,

$$\hat{g}(x) = \int_{-\infty}^{\infty} g(t)\, e(-tx)\, dt$$

$$\|g\|_2 = \left(\int_{-\infty}^{\infty} |g(t)|^2\, dt\right)^{1/2}.$$

2. Non-Degradation of the Minimum Distance For Rates Faster Than the Nyquist Rate

In this section we give a proof, depending upon some auxiliary results proved in the next section, of the result that there is no degradation in the minimum distance between received signals, when signaling at rates somewhat faster than the Nyquist rate. Theorem 2 follows at once by the definition of $I(\delta)$ from the following stronger result:

THEOREM 4. *Let* $Q(\theta) = \sum_{0 \leqslant k \leqslant n} \varepsilon_k e^{2\pi i m_k \theta}$ *where* $0 = m_0 < m_1 < \cdots, m_k$ *are natural numbers, and* $\varepsilon_k = \pm 1$. *Excluding the trivial case of* $Q(\theta) = \pm 1$,

(a) *If the minimal gap between consecutive* m_k *is at least two then*

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} |Q(\theta)|^2\, d\theta \geqslant 1$$

for $0.393 \cdots \leqslant \delta \leqslant 0.5$.

(b) *If the minimal gap between consecutive* m_k *is one and the first place where the gap occurs the coefficients corresponding to the exponentials with consecutive* m_k *have the same sign, then*

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} |Q(\theta)|^2\, d\theta \geqslant 1$$

for $0.38 \cdots \leqslant \delta \leqslant 0.5$.

(c) *If the minimal gap between consecutive m_k is one and the first place where the gap occurs the coefficients corresponding to the exponentials with consecutive m_k have opposite signs then*

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} |Q(\theta)|^2\, d\theta \geqslant 1$$

for $0.4975 \cdots \leqslant \delta \leqslant 0.5$.

It is easy to see numerically, as mentioned in the Introduction, that [5]

$$\frac{1}{2\delta}\int_{\delta}^{\delta} |R(\theta)|^2\, d\theta < 1$$

for $\delta \leqslant 0.4$ where $R(\theta) = 1 + \sum_{j=1}^{7} (-1)^j e^{2\pi i j\theta}$. Thus (a) and (b) of Theorem 4 give a better than expected answer and it is only (c) which does not give as good an answer. It is also quite surprising that there is a distinction between (b) and (c), which is real in view of the numerical example above.

Proof (*Theorem* 4). First we consider part (a). If $Q(\theta)$ has at least K non-zero terms than as a consequence of a theorem of Ingham [4] it follows that

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} |Q(\theta)|^2\, d\theta \geqslant \frac{K}{3}\left(2 - \frac{1}{2\delta}\right).$$

Thus if $K \geqslant 5$ and $\delta \geqslant 5/14$ we are done. On the other hand assume that $Q(\theta)$ has at most four non-zero terms. It is shown in Lemma 5 that if a polynomial $P(\theta)$ has exactly n non-zero terms,

$$P(\theta) = \sum_{i=1}^{n} \varepsilon_{k_i} e(k_i \theta), \qquad \varepsilon_{k_i} = \pm 1$$

then

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} |P(\theta)|^2\, d\theta \geqslant B_n(\delta) = n\left(1 - (n-1)\,\frac{\sin 2\pi\delta}{2\pi\delta}\right).$$

Thus,

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} |Q(\theta)|^2\, d\theta \geqslant \min(B_2(\delta), B_3(\delta), B_4(\delta))$$

$$\geqslant 1$$

for $\delta \geqslant 0.393\cdots$ which proves part (a). We turn to parts (b) and (c). Let $Q(\theta)=\sum_{k=0}^{n} \varepsilon_k e(m_k\theta)$ where $0=m_0<m_1<\cdots$ and $\varepsilon_k=\pm 1$. Let k_0 be the minimal k such that $m_{k+1}-m_k=1$ and let $P(\theta)=\varepsilon_{k_0} e(-m_{k_0}\theta)\, Q(\theta)$. Clearly,

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} |P(\theta)|^2\, d\theta=\frac{1}{2\delta}\int_{-\delta}^{\delta} |Q(\theta)|^2\, d\theta \tag{1}$$

and $P(\theta)$ has the form

$$P(\theta)=1+\varepsilon e(\theta)+\sum_{k\neq 0} \varepsilon_k e(n_k\theta), \tag{2}$$

where ε, $\varepsilon_k=\pm 1$, $n_k \geqslant 2$ for $k \geqslant 1$, $n_k \leqslant -2$ for $k \leqslant -1$, $n_{k+1}-n_k \geqslant 1$ for $k \geqslant 1$, and $n_k-n_{k-1} \geqslant 2$ for $k \leqslant -1$. This is simply by the minimality of k_0, and by (1) we need to only get lower bounds on $P(\theta)$. By assumption in part (b) we have $\varepsilon=1$ in (2) and in part (c) we have $\varepsilon=-1$ in (2). Part (b) follows from Lemma 6.

To see how part (b) follows from this, note that

$$\left(\frac{1}{2\delta}\int_{-\delta}^{\delta} |P(\theta)|^2\, d\theta\right)^{1/2} \geqslant f(\delta),$$

where $f(\delta)$ is the estimate obtained in Lemma 6. Since the series defining $f(\delta)$ converges uniformly, it follows that $f(\delta)$ is continuous. Moreover $f(1/2)=\sqrt{2}$. By the intermediate value theorem there is a $\delta_0<1/2$ such that $f(\delta)\geqslant 1$ for $\delta_0 \leqslant \delta \leqslant 1/2$. A numerical analysis then shows that $\delta_0=0.38\cdots$. Part (c) follows by a similar analysis from Lemma 7. This finishes the proof of Theorem 4. ∎

Finally we prove the extremal form of Theorem 2, namely Theorem 3:

Proof (*Theorem* 3). With the notation as in the statement of Theorem 3, first assume that the minimal gap between non-zero terms in $Q(\theta)$ is at least two. It should be clear that this case can be handled in exactly the same manner as in the proof of Theorem 4(a). If the minimal gap between non-zero terms in $Q(\theta)$ is exactly one, then we may reduce to a polynomial $P(\theta)$ of the form in (2) exactly as in Theorem 4.

For $P(\theta)$ in (2), with $\varepsilon=1$, we have the estimate in Lemma 6. Clearly this estimate implies the result since the estimate goes to $\sqrt{2}$ as δ goes to $1/2$. For $P(\theta)$ in (2) with $\varepsilon=-1$, we need an appropriate analog of Lemma 7, which is provided by Lemma 10 (also see Lemma 9 and the paragraph preceding it). ∎

3. Proof of Auxiliary Results

In this section we prove the auxiliary results needed to prove the theorems in the previous section.

LEMMA 5. $(1/2\delta)\int_{-\delta}^{\delta}|\sum_{i\leqslant n}\varepsilon_{k_i}e(k_i\theta)|^2\,d\theta\geqslant n(1-(n-1)(\sin 2\pi\delta/2\pi\delta))$, *for any* $\varepsilon_{k_i}=\pm 1$.

Proof. Fix $1\leqslant k_1<\cdots<k_n$. We have

$$\frac{1}{2\delta}\int_{-\delta}^{\delta}\left|\sum_{i\leqslant n}\varepsilon_{k_i}e(k_i\theta)\right|^2 d\theta=n+\sum_{\substack{i\neq j\\ i,j\leqslant n}}\varepsilon_{k_i}\varepsilon_{k_j}\frac{\sin\lambda(k_i-k_j)}{\lambda(k_i-k_j)},$$

where $\lambda=2\pi\delta$. Then,

$$\left|\sum_{\substack{i,j\leqslant n\\ i\neq j}}\varepsilon_{k_i}\varepsilon_{k_j}\frac{\sin\lambda(k_i-k_j)}{\lambda(k_i-k_j)}\right|$$

$$\leqslant\sum_{i\leqslant n}\sum_{\substack{j\leqslant n\\ j\neq i}}\frac{|\sin\lambda(k_i-k_j)|}{\lambda\,|k_i-k_j|}$$

$$\leqslant\frac{n}{\lambda}\max_{i\leqslant n}\sum_{\substack{j\leqslant n\\ j\neq i}}|\sin\lambda|$$

(using the inequality $|\sin nx|\leqslant|n|\;|\sin x|$ for integer n)

$$=\frac{n}{\lambda}(n-1)\,|\sin\lambda|,$$

which completes the proof. ∎

The next two lemmas give the estimates needed in parts (b) and (c) of Theorem 4, respectively. They follow at once from Lemma 8, which may be regarded as the basic estimate. We show how they follow from Lemma 8 and then give a proof of Lemma 8.

LEMMA 6. *Let* $\varepsilon=1$ in (2). *Then for* $P(\theta)$ *as in* (2) *and for* $\delta>1/4$,

$$\left(\frac{1}{2\delta}\int_{-\delta}^{\delta}|P(\theta)|^2\,d\theta\right)^{1/2}\geqslant\frac{1}{\sqrt{2}}\left(\left|1+\frac{\sin 2\pi\delta}{\pi}\left(\frac{1}{2\delta}-\frac{1}{2\delta-1}\right)\right|\right.$$
$$\left.-\sum_{k\geqslant 2}\frac{|\sin 2\pi\,\delta k|}{\pi}\left(\frac{1}{2\,\delta k-1}-\frac{1}{2\,\delta k+1}\right)\right).$$

Proof. Set $a_0 = 1$, $a_1 = -1$, and all other $a_n = 0$ in the result of Lemma 8. We obtain

$$\left(\frac{1}{2\delta}\int_{-\delta}^{\delta} |P(\theta)|^2\, d\theta\right)^{1/2} \geqslant \frac{1}{\sqrt{2}}\left(\left|1 + \frac{\sin 2\pi\delta}{\pi}\left(\frac{1}{2\delta} - \frac{1}{2\delta - 1}\right)\right| - \sum_{k \neq 0} \frac{|\sin 2\pi\,\delta n_k|}{\pi}\left|\frac{1}{2\,\delta n_k} - \frac{1}{2\,\delta n_k - 1}\right|\right).$$

Since $n_{k+1} - n_k \geqslant 1$ for $k \geqslant 1$ and $n_k - n_{k-1} \geqslant 2$ for $k < 0$, it follows that for $\delta > 1/4$,

$$\begin{aligned}\left(\frac{1}{2\delta}\int_{-\delta}^{\delta} |P(\theta)|^2\, d\theta\right)^{1/2} &\geqslant \frac{1}{\sqrt{2}}\left(\left|1 + \frac{\sin 2\pi\delta}{\pi}\left(\frac{1}{2\delta} - \frac{1}{2\delta - 1}\right)\right|\right. \\ &\quad - \sum_{k \geqslant 2} \frac{|\sin 2\pi\,\delta k|}{\pi}\left(\frac{1}{2\,\delta k - 1} - \frac{1}{2\,\delta k}\right) \\ &\quad \left. - \sum_{k \geqslant 2} \frac{|\sin 2\pi\,\delta k|}{\pi}\left(-\frac{1}{2\,\delta k + 1} + \frac{1}{2\,\delta k}\right)\right) \\ &= \frac{1}{\sqrt{2}}\left(\left|1 + \frac{\sin 2\pi\delta}{\pi}\left(\frac{1}{2\delta} - \frac{1}{2\delta - 1}\right)\right|\right. \\ &\quad \left. - \sum_{k \geqslant 2} \frac{|\sin 2\pi\,\delta k|}{\pi}\left(\frac{1}{2\,\delta k - 1} - \frac{1}{2\,\delta k + 1}\right)\right).\end{aligned}$$

Note that the series in question converge uniformly in δ and also absolutely (thus rearrangements are allowed). ∎

LEMMA 7. *Let* $\varepsilon = -1$ *in* (2). *Then for* $P(\theta)$ *as in* (2) *and* $\delta > 5/12$,

$$\begin{aligned}\left(\frac{1}{2\delta}\int_{-\delta}^{\delta} |P(\theta)|^2\, d\theta\right)^{1/2} &\geqslant \frac{\sqrt{3}}{\sqrt{10}}\left(\left|1 - \frac{\sin 2\pi\delta}{\pi}\left(\frac{1}{2\delta} + \frac{1}{2\delta - 1}\right.\right.\right. \\ &\quad \left.\left. - \frac{2}{3}\left(\frac{1}{2\delta + b_1} + \frac{1}{2\delta + b_2} + \frac{1}{2\delta + b_3}\right)\right)\right| \\ &\quad - \sum_{k \leqslant -6,\, k \geqslant 2} \frac{|\sin 2\pi\,\delta k|}{\pi}\left|\frac{1}{2\,\delta k} + \frac{1}{2\,\delta k - 1}\right. \\ &\quad \left. - \frac{2}{3}\left(\frac{1}{2\,\delta k + b_1} + \frac{1}{2\,\delta k + b_2} + \frac{1}{2\,\delta k + b_3}\right)\right| \\ &\quad - \sum_{k \in A} \frac{|\sin 2\pi\,\delta k|}{\pi}\left|\frac{1}{2\,\delta k} + \frac{1}{2\,\delta k - 1}\right. \\ &\quad \left.\left. - \frac{2}{3}\left(\frac{1}{2\,\delta k + b_1} + \frac{1}{2\,\delta k + b_2} + \frac{1}{2\,\delta k + b_3}\right)\right|\right),\end{aligned}$$

where

(a) $b_1=1$, $b_2=2$, $b_3=3$, and $A=\{-4,-5\}$ if $|n_{-1}|\geqslant 4$.

(b) $b_1=1$, $b_2=2$, $b_3=4$, and $A=\{-3,-5\}$ if $n_{-1}=-3$.

(c) $b_1=1$, $b_2=3$, $b_3=4$, and $A=\{-2,-5\}$ if $n_{-1}=-2$, $|n_{-2}|\geqslant 5$.

(d) $b_1=1$, $b_2=3$, $b_3=5$, and $A=\{-2,-4\}$ if $n_{-1}=-2$, $n_{-2}=-4$.

Proof. First note that all the series in question converge uniformly. This is because

$$\frac{1}{x}+\frac{1}{x-1}-\frac{2}{3}\left(\frac{1}{x+b_1}+\frac{1}{x+b_2}+\frac{1}{x+b_3}\right)=\frac{P(x)}{Q(x)},$$

where $Q(x)$ is a polynomial of degree 5 and $P(x)$ is a polynomial of degree 3 and so,

$$\sum_{k\leqslant -6,\, k\geqslant 2}\frac{|P(2k\delta)|}{|Q(2k\delta)|}$$

is finite. Also note since $\delta>5/12$ none of the denominators, $2\,\delta k$, $2\,\delta k-1$, $2\,\delta k+b_1$, $2\,\delta k+b_2$, $2\,\delta k+b_3$ vanish. We now show the estimate (a). The others are obtained in an analogous manner. Set $\varepsilon=-1$, $a_0=1$, $a_1=1$, $a_{-1}=-2/3$, $a_{-2}=-2/3$, $a_{-3}=-2/3$, and all other $a_n=0$ in the estimate for Lemma 8. We get

$$\begin{aligned}\left(\frac{1}{2\delta}\int_{-\delta}^{\delta}|P|^2\,d\theta\right)^{1/2}\geqslant\frac{\sqrt{3}}{\sqrt{10}}&\left(\left|1-\frac{\sin 2\pi\delta}{\pi}\left(\frac{1}{2\delta}+\frac{1}{2\delta-1}\right.\right.\right.\\&\left.\left.-\frac{2}{3}\left(\frac{1}{2\delta+1}+\frac{1}{2\delta+2}+\frac{1}{2\delta+3}\right)\right)\right|\\&-\sum_{k\neq 0}\frac{|\sin 2\pi\,\delta n_k|}{\pi}\left|\frac{1}{2\,\delta n_k}+\frac{1}{2\,\delta n_k-1}\right.\\&\left.\left.-\frac{2}{3}\left(\frac{1}{2\,\delta n_k+1}+\frac{1}{2\,\delta n_k+2}+\frac{1}{2\,\delta n_k+3}\right)\right|\right)\\\geqslant\frac{\sqrt{3}}{\sqrt{10}}&\left(\left|1-\frac{\sin 2\pi\delta}{\pi}\left(\frac{1}{2\delta}+\frac{1}{2\delta-1}\right.\right.\right.\\&\left.\left.-\frac{2}{3}\left(\frac{1}{2\delta+1}+\frac{1}{2\delta+2}+\frac{1}{2\delta+3}\right)\right)\right|\\&-\sum_{k\leqslant -4,\,k\geqslant 2}\frac{|\sin 2\pi\,\delta k|}{\pi}\left|\frac{1}{2\,\delta k}+\frac{1}{2\,\delta k-1}\right.\\&\left.\left.-\frac{2}{3}\left(\frac{1}{2\,\delta k+1}+\frac{1}{2\,\delta k+2}+\frac{1}{2\,\delta k+3}\right)\right|\right)\end{aligned}$$

since $|n_{-k}|\geqslant 4$ for $k\geqslant 1$ and $n_k\geqslant 2$ for $k\geqslant 1$ if we assume that $|n_{-1}|\geqslant 4$.

Note that $|n_{-1}| \geqslant 4$ also ensures $2\delta n_k + i \neq 0$ for $i = 0, -1, 1, 2, 3$ and any k. ▌

LEMMA 8. *Let* $P(\theta) = 1 + \varepsilon e(\theta) + \sum_{k \geqslant 1} \varepsilon_k e(n_k \theta) + \sum_{k < 0} \varepsilon_k e(n_k \theta)$ *be a polynomial as in* (2). *Then, for any complex numbers* a_n *with* $\sum_{-\infty}^{\infty} |a_n|^2 < +\infty$,

$$\left(\frac{1}{2\delta}\int_{-\delta}^{\delta} |P|^2\, d\theta\right)^{1/2} \geqslant \left(\sum_{-\infty}^{\infty} |a_n|^2\right)^{-1/2} \left(\left|a_0 + \frac{\varepsilon \sin 2\pi\delta}{\pi} \sum_{-\infty}^{\infty} \frac{a_n}{2\delta - n}\right| - \sum_{k \neq 0} \frac{|\sin 2\pi\, \delta n_k|}{\pi} \left|\sum_{-\infty}^{\infty} \frac{a_n}{2\,\delta n_k - n}\right|\right).$$

Proof. Let $g \in L_2(-\infty, \infty)$ and suppose that $\hat{g}$ is supported in $[-1/2, 1/2]$. Let $\eta = 2\delta$ where $0 < \delta \leqslant 1/2$ and let $g_\eta(x) = g(\eta x)$. Then $\hat{g}_\eta(x) = (1/\eta)\, \hat{g}(x/\eta)$ so that the support of $\hat{g}_\eta$ is $[-\delta, \delta]$. Moreover a simple computation shows

$$\|g_\eta\|_2^2 = 1/\eta\, \|g\|_2^2.$$

Thus $\hat{g}_\eta \in L_2(-\delta, \delta)$ and so upon applying the Cauchy–Schwartz inequality, Plancherel's Theorem, the triangle inequality, and the inversion theorem,

$$\begin{aligned}
&\|g\|_2 \left(\frac{1}{\eta}\int_{-\delta}^{\delta} |P(\theta)|^2\, d\theta\right)^{1/2} \\
&= \|\hat{g}_\eta\|_2 \left(\int_{-\delta}^{\delta} |P(\theta)|^2\, d\theta\right)^{1/2} \\
&\geqslant \left|\int_{-\delta}^{\delta} \left(\hat{g}_\eta(\theta) + \varepsilon \hat{g}_\eta(\theta)\, e(\theta) + \sum_{k \neq 0} \varepsilon_k\, \hat{g}_\eta(\theta)\, e(n_k \theta)\right) d\theta\right| \\
&\geqslant \left|\int_{-\delta}^{\delta} (\hat{g}_\eta(\theta) + \varepsilon \hat{g}_\eta(\theta) e(\theta))\, d\theta\right| - \sum_{k \neq 0} \left|\int_{-\delta}^{\delta} \hat{g}_\eta(\theta)\, e(n_k \theta)\, d\theta\right| \\
&= |g(0) + \varepsilon g(\eta)| - \sum_{k \neq 0} |g(\eta n_k)|.
\end{aligned}$$

Therefore for any $g \in L_2(-\infty, \infty)$ and $\hat{g}$ supported in $[-1/2, 1/2]$,

$$\left(\frac{1}{2\delta}\int_{-\delta}^{\delta} |P(\theta)|^2\, d\theta\right)^{1/2} \geqslant \frac{1}{\|g\|_2} \left(|g(0) + \varepsilon g(\eta)| - \sum_{k \neq 0} |g(\eta n_k)|\right).$$

By the Paley–Weiner Theorem [7] such a g may be identified with an entire function $g(z)$ with $|g(z)| \leqslant Ae^{\pi|z|}$ for some $A > 0$, and in turn such a $g(z)$ may be written as

$$g(z) = \frac{\sin \pi z}{\pi} \sum_{-\infty}^{\infty} \frac{(-1)^n g(n)}{z-n}$$

by Hardy's Theorem [7]. It follows that for any $(a_n)_{n=-\infty}^{\infty}$ with, $\sum_{-\infty}^{\infty} |a_n|^2 < +\infty$ we may define a $g(z)$ by letting $g(n) = (-1)^n a_n$ and then

$$\left(\frac{1}{2\delta} \int_{-\delta}^{\delta} |P|^2 \, d\theta\right)^{1/2} \geqslant \left(\sum_{-\infty}^{\infty} |a_n|^2\right)^{-1/2} \left(\left|a_0 + \varepsilon \frac{\sin 2\pi\delta}{\pi} \sum_{-\infty}^{\infty} \frac{a_n}{2\delta - n}\right| - \sum_{k \neq 0} \frac{|\sin 2\pi \delta n_k|}{\pi} \left|\sum_{-\infty}^{\infty} \frac{a_n}{2\delta n_k - n}\right|\right);$$

since $\|g\|_2^2 = \sum_{-\infty}^{\infty} |g(n)|^2$. ∎

The next two lemmas are needed for Theorem 3. Note that in (2) the negative n_k in $P(\theta)$ satisfy $n_k - n_{k-1} \geqslant 2$ for $k < 0$ and $n_{-1} \leqslant -2$. In stating Lemma 10 below, we will be interested in finite sequences $(b_k)_{1 \leqslant k \leqslant l}$ of positive integers. Given (n_k) as in (2) we define the corresponding sequence $(b_k)_{1 \leqslant k \leqslant l}$ recursively by setting $b_1 = 1$ and given b_k, b_{k+1} is the next positive integer which is not $|n_j|$ for some $j < 0$. Note that such a sequence b_k satisfies $1 \leqslant b_{k+1} - b_k \leqslant 2$. The following trivial lemma generates such sequences, and the proof is left to the reader.

LEMMA 9. *There are 2^{l-1} sequences $(b_k)_{1 \leqslant k \leqslant l}$ with $b_1 = 1$ and $1 \leqslant b_{k+1} - b_k \leqslant 2$. They may be generated by the following procedure: Given any $(\varepsilon_k)_{k=1}^{l-1}$ with $\varepsilon_k = 0, 1$ let the corresponding $(b_k)_{k=1}^{l}$ be $b_1 = 1$ and $b_{k+1} = b_k + 1$ in case $\varepsilon_k = 0$ and $b_{k+1} = b_k + 2$ in case $\varepsilon_k = 1$.*

We may now get the estimate we want.

LEMMA 10. *Let $\varepsilon = -1$ and let $P(\theta)$ be as in (2). Then for any $l \geqslant 1$ and $1/2 - 1/4l < \delta \leqslant 1/2$,*

$$\left(\frac{1}{2\delta} \int_{-\delta}^{\delta} |P(\theta)|^2 \, d\theta\right)^{1/2}$$

$$\geqslant \min \frac{1}{(2+4/l)^{1/2}} \left(\left|1 - \frac{\sin 2\pi\delta}{\pi} \left(\frac{1}{2\delta} + \frac{1}{2\delta - 1} - \frac{2}{l} \sum_{j=1}^{l} \frac{1}{2\delta + b_j}\right)\right| - \sum_{\substack{|k| \geqslant 2 \\ k \notin \{-b_1, -b_2, \ldots, -b_l\}}} \frac{|\sin 2\pi \delta k|}{\pi} \left|\frac{1}{2\delta k} + \frac{1}{2\delta k - 1} - \frac{2}{l} \sum_{j=1}^{l} \frac{1}{2\delta k + b_j}\right|\right),$$

where the minimum is over the $(b_k)_{k=1}^{l}$ *in Lemma* 9. *Thus given* $1 \leqslant M < \sqrt{2}$ *there is a* $\delta_0 < 1/2$ *such that for* $\delta_0 \leqslant \delta \leqslant 1/2$,

$$\left(\frac{1}{2\delta}\int_{-\delta}^{\delta} |P(\theta)|^2\, d\theta\right)^{1/2} \geqslant M.$$

Proof. As in the proof of Lemma 7 the series in question converge uniformly. Let $P(\theta)$ be as in Theorem 2. Let $(b_k)_{k=1}^{l}$ be such that the b_k lie in between the consecutive terms of $|n_j|$ for $j<0$ with $b_1 = 1$, $1 \leqslant b_{k+1} - b_k \leqslant 2$. In the estimate of Lemma 8 set $a_0 = 1$, $a_1 = 1$, $a_{-k} = -2/l$ for $k = b_1, ..., b_l$, and all other $a_n = 0$. We get

$$\begin{aligned}
&\left(\frac{1}{2\delta}\int_{-\delta}^{\delta} |P(\theta)|^2\, d\theta\right)^{1/2} \\
&\quad \geqslant \frac{1}{(2+4/l)^{1/2}}\left(\left|1 - \frac{\sin 2\pi\delta}{\pi}\left(\frac{1}{2\delta} + \frac{1}{2\delta - 1} - \frac{2}{l}\sum_{j=1}^{l}\frac{1}{2\delta + b_j}\right)\right|\right. \\
&\qquad \left. - \sum_{k \neq 0} \frac{|\sin 2\pi\,\delta n_k|}{\pi}\left|\frac{1}{2\,\delta n_k} + \frac{1}{2\,\delta n_k - 1} - \frac{2}{l}\sum_{j=1}^{l}\frac{1}{2\,\delta n_k + b_j}\right|\right) \\
&\quad \geqslant \frac{1}{(2+4/l)^{1/2}}\left(\left|1 - \frac{\sin 2\pi\delta}{\pi}\left(\frac{1}{2\delta} + \frac{1}{2\delta - 1} - \frac{2}{l}\sum_{j=1}^{l}\frac{1}{2\delta + b_j}\right)\right|\right. \\
&\qquad \left. - \sum_{\substack{|k| \geqslant 2 \\ k \notin \{-b_1, ..., b_l\}}} \frac{|\sin 2\pi\,\delta k|}{\pi}\left|\frac{1}{2\,\delta k} + \frac{1}{2\,\delta k - 1} - \frac{2}{l}\sum_{j=1}^{l}\frac{1}{2\,\delta k + b_j}\right|\right)
\end{aligned}$$

since $n_k \geqslant 2$ for $k \geqslant 1$ and $n_k \leqslant -2$ for $k \leqslant -1$. Moreover note that none of the denominators $2\,\delta n_k + b_j$ are zero by the choice of b_j for $\delta > 1/2 - 1/4l$. The last part of the statement of the theorem is obvious by continuity and by making l sufficiently large. ∎

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Note added in proof. Landau and Mazo [9] have independently obtained the stronger version of Theorem 2 (see [3]) mentioned in the introduction.

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